Int. J. Heat Mass Transfer. Vol. 6, pp. 703–718. Pergamon Press 1963. Printed in Great Britain.

TRANSIENT COOLING OF A HEATED ENCLOSURE

A. W. PRATT and E. F. BALL
Building Research Station, Garston, Watford, Herts.

(*Received* 31 *January* 1963)

Abstract—A theoretical examination is made of the transient change in the air temperature inside an enclosure, in which heat is produced at a uniform rate, following a step-function change in the outside temperature. The influences of ventilation, internal heat storage, and the thermal properties of the walls of the enclosure upon the rate of change of inside temperature are compared by numerical calculation for a few selected cases. It is shown that under certain conditions the ratio (heat stored/rate of heat loss from the enclosure) in the steady state is the parameter which determines the inside temperature variation.

NOMENCLATURE

θ_i, inside air temperature, degF;
θ_o, outside air temperature, degF;
$\theta(x)$, temperature at position x in wall, degF;
θ_{is}, temperature of internal mass, degF;
h_i, inside wall surface heat transfer coefficient, Btu/ft^{2}h degF;
h_o, outside wall surface heat transfer coefficient, Btu/ft^{2}h degF;
h_s, heat transfer coefficient at surface of internal mass expressed per unit area of enclosure wall, Btu/ft^{2}h degF;
c, thermal capacity of internal mass per unit area of wall, Btu/ft^2 degF;
v, ventilation heat loss rate per unit temperature difference and per unit area of enclosure wall, Btu/ft^{2}h degF;
q, heat input per unit area of wall;
H, conduction heat flux per unit temperature difference, Btu/ft^{2}h degF;
W, heat stored per unit area of wall, Btu/ft^2;
T, thermal time constant, h;
k, thermal conductivity, Btu/ft h degF;
K, thermal diffusivity, ft^2/h;
ρ, density, lb/ft^3;
s, specific heat, Btu/lb degF;
l, thickness of wall, ft;
x, position in wall, ft;
t, time, h;
$a_n, n = 1, 2, 3, \ldots$, successive roots of relevant function;
$[a_1][a_2]$, approximations to a_1, a_2;
$\xi = x/l$;
$\tau = Kt/l^2$;
$B_i = h_i l/k$;
$B_o = h_o l/k$;
$B_s = h_s l/k$;
$Q = ql/k$;
$V = vl/k$;
$C = cK/lk$;
$A = 1 + 1/B_i + 1/B_o$;
$\beta = [B_s(B_i + V)]/[C(B_s + B_i + V)]$.

Additional nomenclature used in Appendix 1

$\theta_1(x)$, temperature at position x in outer leaf of cavity wall ($0 < x < l_1$), degF;
$\theta_2(x)$, temperature at position x in inner leaf of cavity wall ($l_1 < x < l_2$), degF;
h, coefficient of heat transfer across cavity, Btu/ft^{2}h degF;
l_1, thickness of outer leaf, ft;
$(l_2 - l_1)$, thickness of inner leaf, ft;
k_1, thermal conductivity of outer leaf, Btu/ft h degF;
k_2, thermal conductivity of inner leaf, Btu/ft h degF;
K_1, thermal diffusivity of outer leaf, ft^2/h;
K_2, thermal diffusivity of inner leaf, ft^2/h;
ρ_1, density of outer leaf, lb/ft^3;
ρ_2, density of inner leaf, lb/ft^3;
s_1, specific heat of outer leaf, Btu/lb degF;
s_2, specific heat of inner leaf, Btu/lb degF;
$R_{i.o}$, air-to-air thermal resistance of cavity wall, ft^{2}h degF/Btu;
$a_o = h_o\sqrt{(K_1)}/k_1$;
$a = h\sqrt{(K_1)}/k_1$;

$a_1 = h\sqrt{(K_2)}/k_2$;
$a_2 = h_i\sqrt{(K_2)}/k$;
$R_1(x) = 1/h_o + x/k_1\ (0 \leq x \leq l_1)$, ft²h degF/Btu;
$R_2(x) = 1/h_o + l_1/k_1 + 1/h + (x - l_1)/k_2$
$(l_1 \leq x \leq l_2)$ ft²h degF/Btu;
$b_1 = l_1/\sqrt{K_1}$;
$b_2 = (l_2 - l_1)/\sqrt{K_2}$.

INTRODUCTION

THE TRANSIENT flow of heat in a structure assumes a variety of different forms that may have a practical interest. The present analysis for example derives from a consideration of the cooling rate of buildings as it affects the selection of basic design temperatures for space-heating installations. The present formulation is hypothetical; it is intended to be of general interest without reference to any particular practical situation.

The rate at which an enclosure responds thermally to a change applied in the ambient temperature will depend upon the thermal capacities of the enclosing wall and the interior mass and, if ventilated, on the rate of air change. In this paper, exact analytical solutions are obtained and evaluated numerically to demonstrate the interplay and relative importance of the various factors for a few selected cases. The analysis is an improvement on less rigorous methods used in previously published work which deals primarily with the application of the problem to buildings and considers heat transfer in the external sections only [1, 2, 3].

FORMULATION OF THE PROBLEM

The situation considered in this paper and illustrated in Fig. 1, is the time-variation of the air temperature within an enclosure following a sudden drop from θ_o to zero in the ambient temperature, which is assumed to be uniformly distributed over the external boundary for all values of time. In the initial steady state condition, and at all subsequent values of time, heat is added directly to the air inside the enclosure at a constant rate q. Heat is lost from the enclosure by conduction; and further heat is lost by exchange of inside and outside air, that is by ventilation, at a constant rate v per unit difference in air temperature; otherwise the heat capacity of the enclosed air is assumed to be negligibly small. The enclosure contains internal mass, of thermal capacity c, whose temperature is assumed to be uniform throughout for all $t > 0$ and initially equal to that of the internal air. The quantities q, c and v are defined per unit area of the enclosure wall. For all values of time the surfaces of the internal mass at temperature $\theta_{is}(t)$, and both surfaces of the enclosure wall at temperatures $\theta(l, t)$ (inside surface), and $\theta(0, t)$ (outside surface), exchange heat with the contiguous air at h_s, h_i and h_o times their respective temperature differences per unit area of the wall and per unit time. The analysis considers the enclosure walls as a slab of infinite extent bounded by parallel planes distance l apart, with constant thermal properties (conductivity k, diffusivity K), that is, the conduction heat transfer is assumed to be uni-directional along the x axis and perpendicular to the surfaces.

Under these conditions an expression for the inside air temperature, $\theta_i(t)$, is required that satisfies the problem formulated mathematically as follows,

$$K\frac{\partial^2\theta(x, t)}{\partial x^2} = \frac{\partial\theta(x, t)}{\partial t}, \ (0 < x < l, \ t > 0) \tag{1}$$

$$k\frac{\partial\theta(x, t)}{\partial x} = h_o\theta(x, t), \ (x = 0, \ t > 0) \tag{2}$$

$$-k\frac{\partial\theta(x, t)}{\partial x} = h_i[\theta_i(t) - \theta(x, t)], \ (x = l, \ t > 0) \tag{3}$$

$$q = h_i[\theta_i(t) - \theta(x, t)] + v\theta_i(t) - h_s[\theta_{is}(t) - \theta_i(t)], \ (x = l, \ t > 0) \tag{4}$$

$$c\frac{\mathrm{d}\theta_{is}(t)}{\mathrm{d}t} = -h_s[\theta_{is}(t) - \theta_i(t)], \ (t > 0). \tag{5}$$

Initially the system is in steady state so that

$$\theta(x, 0) = \theta_o + \frac{(1/h_o + x/k)(\theta_i - \theta_o)}{(1/h_o + l/k + 1/h_i)}.$$

Now

$$(\theta_i - \theta_o) = q/[v + (1/h_o + l/k + 1/h_i)^{-1}].$$

Therefore, by substitution

$$\theta(x, 0) = \theta_o + \frac{(1/h_o + x/k)}{(1/h_o + l/k + 1/h_i)} \frac{q}{[v + (1/h_o + l/k + 1/h_i)^{-1}]}$$

$$= \theta_o + \frac{(1/h_o + x/k)q}{1 + v(1/h_o + l/k + 1/h_i)},$$

$$(0 < x < l,\ t = 0). \quad (6)$$

Also $\theta_{is} = \theta_i$, $(t = 0)$.

Making the substitutions

$$\xi = x/l,\ \tau = Kt/l^2,\ B_i = h_i l/k,\ B_o = h_o l/k,$$
$$B_s = h_s l/k,\ Q = ql/k,\ C = cK/lk,\ V = vl/k,$$

equations (1)–(6) are written more conveniently as follows:

$$\frac{\partial^2\theta(\xi, \tau)}{\partial\xi^2} = \frac{\partial\theta(\xi, \tau)}{\partial\tau},\ (0 < \xi < 1,\ \tau > 0) \quad (7)$$

$$\frac{\partial\theta(\xi, \tau)}{\partial\xi} = B_o\theta(\xi, \tau),\ (\xi = 0,\ \tau > 0) \quad (8)$$

$$\frac{\partial\theta(\xi, \tau)}{\partial\xi} = B_i[\theta_i(\tau) - \theta(\xi, \tau)],\ (\xi = 1,\ \tau > 0) \quad (9)$$

$$Q = B_i[\theta_i(\tau) - \theta(\xi, \tau)] + V\theta_i(\tau) - B_s[\theta_{is}(\tau) - \theta_i(\tau)],\ (\xi = 1,\ \tau > 0) \quad (10)$$

$$C\frac{d\theta_{is}(\tau)}{d\tau} = -B_s[\theta_{is}(\tau) - \theta_i(\tau)],\ (\tau > 0) \quad (11)$$

$$\theta(\xi, 0) = \theta_o + \frac{Q\xi}{1 + VA} + \frac{Q/B_o}{1 + VA} \quad (12)$$

where $A = 1 + 1/B_i + 1/B_o$.

The equations in dimensionless form are solved by a routine application of the Laplace transform. Multiplying (7) through (11) by $e^{-p\tau}$ and integrating with respect to τ between the limits $0, \infty$ it is found that the function $\bar{\theta}(\xi, p)$ satisfies the following transformed differential equation and boundary conditions,

$$\frac{d^2\bar{\theta}(\xi, p)}{d\xi^2} - p\bar{\theta}(\xi, p) + \frac{Q\xi}{1 + VA} + \frac{Q/B_o}{1 + VA} + \theta_o = 0,\ (0 < \xi < 1)\ (\tau > 0) \quad (13)$$

$$\frac{d\bar{\theta}(\xi, p)}{d\xi} = B_o\bar{\theta}(\xi, p),\ (\xi = 0,\ \tau > 0) \quad (14)$$

$$\frac{d\bar{\theta}(\xi, p)}{d\xi} = B_i[\bar{\theta}_i(p) - \bar{\theta}(\xi, p)],\ (\xi = 1,\ \tau > 0) \quad (15)$$

$$\frac{Q}{p} = B_i[\bar{\theta}_i(p) - \bar{\theta}(\xi, p)] + V\bar{\theta}_i(p) - B_s[\bar{\theta}_{is}(p) - \bar{\theta}_i(p)],\ (\xi = 1,\ \tau > 0) \quad (16)$$

$$Cp\bar{\theta}_{is}(p) = C\left(\theta_o + \frac{QA}{1 + VA}\right) - B_s[\bar{\theta}_{is}(p) - \bar{\theta}_i(p)],\ (\tau > 0) \quad (17)$$

where $\bar{\theta}(\xi, p) = \int_0^\infty e^{-p\tau}\theta(\xi, \tau)\, d\tau$.

The general solution of (13) subject to (14) through (17) is

$$\bar{\theta}(\xi, p) = \frac{1}{p}\left[\theta_o + Q\frac{(\xi + 1/B_o)}{1 + VA}\right] - \frac{1}{\sqrt{(p)}}\frac{\theta_o}{\phi(p)}\left(\frac{VB_iB_o}{p}(Cp + B_s) - \frac{B_o}{p}\{B_i(Cp + B_s)\sqrt{(p)}\sinh\sqrt{(p)} + [V(Cp + B_s) + CpB_s][B_i\cosh\sqrt{(p)} + \sqrt{(p)}\sinh\sqrt{(p)}\}\right)\sinh\sqrt{(p\xi)}$$
$$- \frac{1}{\sqrt{(p)}}\frac{\theta_o}{\phi(p)}\left(\frac{VB_i}{\sqrt{(p)}}(Cp + B_s) + \frac{B_o}{p}\{B_i(Cp + B_s)\sqrt{(p)}\sinh\sqrt{(p)} + [V(Cp + B_s) + CpB_s][B_i\sinh\sqrt{(p)} + \sqrt{(p)}\cosh\sqrt{(p)}]\}\right)\cosh\sqrt{(p)}\xi \quad (18)$$

where

$$\phi(p) = [V(Cp + B_s)(B_i + B_o) + CpB_s(B_i + B_o) + B_iB_o(Cp + B_s)]\cosh\sqrt{(p)} + [(Cp + B_s) + B_sCp + CB_iB_oB_s]\sqrt{(p)}\sinh\sqrt{(p)} + V(Cp + B_s)(p - B_iB_o)(\sinh\sqrt{p})/\sqrt{p}. \quad (19)$$

Equation (18) may be inverted to give the temperature distribution, $\theta(\xi, \tau)$, within the enclosure wall: attention however will be confined to calculating a solution for the inside air temperature $\theta_i(\tau)$. From (17)

$$\bar{\theta}_{is}(p) = \frac{1}{(Cp + B_s)} \left[C \left(\theta_o + \frac{QA}{1 + VA} \right) + B_s \bar{\theta}_i(p) \right]. \tag{20}$$

Substituting the expression (20) for $\bar{\theta}_{is}(p)$ into (16) yields

$$\bar{\theta}_i(p) = \frac{1}{\left(B_i + V + \dfrac{B_sCp}{Cp + B_s}\right)} \left[\frac{Q}{p} + \frac{B_sC}{Cp + B_s} \left(\theta_o + \frac{QA}{1 + VA} \right) \right] + \frac{B_i\bar{\theta}(\xi, p)}{\left(B_i + V + \dfrac{B_sCp}{Cp + B_s}\right)}$$

$$(\xi = 1). \tag{21}$$

Substituting for $[\bar{\theta}(\xi, p)]_{\xi=1}$ into (21) from (18) leads, after rearrangement, to the result

$$\bar{\theta}_i(p) = \frac{QA}{p(1 + VA)} + \frac{\theta_o}{p} \left[\frac{B_i + CpB_s/(Cp + B_s)}{B_i + V + CpB_s/(Cp + B_s)} \right] - \frac{B_iB_o\theta_o}{p\phi(p)}(Cp + B_s)$$
$$- \frac{B_iV\theta_o(Cp + B_s)}{p\phi(p)\left(B_i + V + \dfrac{B_sCp}{Cp + B_s}\right)} \left[B_i \cosh \sqrt{(p)} + \frac{B_o}{\sqrt{(p)}} \sinh \sqrt{(p)} \right]. \tag{22}$$

The air temperature, $\theta_i(\tau)$, is found from its Laplace transform (22) by use of the inversion theorem and contour integration. Thus

$$\theta_i(\tau) = \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} e^{p\tau} \bar{\theta}_i(p) \, \mathrm{d}p, \; (\gamma > 0).$$

The transform $\bar{\theta}_i(p)$ is seen to be an analytic function of p with a simple pole at the origin of the p-plane and the other singularities simple poles at points located on the negative real axis. Carrying out the integration indicated, the solution for the air temperature inside the enclosure is as follows:

$$\theta_i(\tau) = \frac{QA}{1 + VA} + 2B_iB_o\theta_o \sum_{n=1}^{\infty} \frac{(B_s - C\alpha_n^2)}{\alpha_n^2 F(\alpha_n)} \exp(-\alpha_n^2\tau) - \frac{V}{C}\left\{ \frac{CB_s\theta_o e^{-\beta\tau}}{(B_i + B_s + V)(B_i + V)} + B_iB_oB^3\theta_o \right.$$
$$\left. \frac{[(B_i/B_o) \cos \sqrt{(\beta)} + (B_i/\sqrt{\beta}) \sin \sqrt{\beta}]e^{-\beta\tau}}{\beta[\phi(p)]_{p=-\beta} (V + B_i + B_s)^2} \right\} - 2\frac{V}{C} B_iB_o\theta_o$$
$$\sum_{n=1}^{\infty} \frac{(B_s - C\alpha_n^2)^2[(B_i/B_o) \cos \alpha_n + (B_i/\alpha_n) \sin \alpha_n] \exp(-\alpha_n^2\tau)}{\alpha_n^2 F(\alpha_n)(B_i + B_s + V)(\alpha_n^2 - \beta)} \tag{23}$$

where

$$\beta = \frac{B_s(B_i + V)}{C(B_i + B_s + V)}, \quad F(\alpha_n) = 2\left[\frac{\mathrm{d}\phi(p)}{\mathrm{d}p}\right]_{p = -\alpha_n^2} = [B_sC(2B_i + 2B_o + B_iB_o) + B_i(B_s + 2CB_o)$$
$$- C(B_i + B_s)\alpha_n^2] \cos \alpha_n + [B_iB_s[1 + B_o(1 + C)] - [B_s(B_i + B_o + 3)$$
$$+ B_i(B_o + 3)]C\alpha_n^2] (\sin \alpha_n)/\alpha_n + V\{[B_s + C(2B_i + 2B_o + B_iB_o) - C\alpha_n^2 - B_iB_oB_s/\alpha_n^2] \cos \alpha_n$$
$$+ [CB_iB_o + B_s(1 + B_i + B_o) - C(B_i + B_o + 3)\alpha_n^2 + B_iB_oB_s/\alpha_n^2](\sin \alpha_n)/\alpha_n\}$$

and the summation is taken over the positive roots α_n, $n = 1, 2, 3$, etc. of the eigenfunction

$$\alpha \tan \alpha = \frac{C\alpha^4[(V + B_s)(B_i + B_o) + B_iB_o] - B_s\alpha^2[V(B_i + B_o) + B_iB_o]}{C\alpha^4(B_i + B_s + V) - \alpha^2[CB_iB_o(V + B_s) + B_s(V + B_i)] + VB_iB_oB_s}. \tag{24}$$

Numerical evaluation of the exact solution (23) is tedious, largely because of the labour involved in calculating the zeros of $\phi(p)$ using (24). Except for small values of time, however, the roots higher than the second are unlikely to contribute significantly to the computed result.

Approximate values of the zeros α_1^2, α_2^2 may be calculated more easily by expressing $\phi(p)$ as a polynomial in p; thus

$$\phi(p) = B_s[B_iB_o + V(B_i + B_o + B_iB_o)] + \left[B_s\left(B_i + \frac{B_iB_o}{2!}\right) + C[B_s(B_i + B_o + B_iB_o) + B_iB_o]\right.$$
$$\left. + CV(B_i + B_o + B_iB_o) + VB_s\left(1 + \frac{B_i + B_o}{2!} + \frac{B_iB_o}{3!}\right)\right] p + \left\{B_s\left(\frac{B_i}{3!} + \frac{B_iB_o}{4!}\right)\right.$$
$$+ C\left[B_s\left(1 + \frac{B_i + B_o}{2!} + \frac{B_iB_o}{3!}\right) + B_i\left(1 + \frac{B_o}{2!}\right)\right] + CV\left(1 + \frac{B_i + B_o}{2!} + \frac{B_iB_o}{3!}\right)$$
$$\left. + VB_s\left(\frac{1}{3!} + \frac{B_i + B_o}{4!} + \frac{B_iB_o}{5!}\right)\right\} p^2 + \ldots \tag{25}$$

The polynomial (25) has distinct zeros at $p = -\alpha_1^2, -\alpha_2^2, \ldots$; decomposing $\phi(p)$ into linear factors and factorizing out the product $\alpha_1^2\alpha_2^2 \ldots$ the function may be written $\phi(p) = \alpha(1 + p/\alpha_1^2)(1 + p/\alpha_2^2) \ldots$ where α is a constant which, by putting $p = 0$ in (25) is recognized as $\alpha = (1 + VA)B_iB_oB_s$. Writing, therefore, $\phi(p) = (1 + VA)B_iB_oB_s \prod_{r=1}^{\infty} (1 + p/\alpha_r^2)$ the calculation of the residues now takes the form

$$L^{-1}\left[\frac{1}{\phi(p)}\right] = \sum_{r=1}^{\infty} \frac{\exp - (\alpha_r^2\tau)}{(1 + VA)B_iB_oB_s \prod_{s=1}^{\infty}{}' (1 + p/\alpha_s^2)_{p=-\alpha_r^2}} \tag{26}$$

where L^{-1} denotes the Laplace inversion and the prime in Π' indicates that the term given by $s = r$ is omitted from the product.

For numerical purposes experience suggests that the residues contributed by the infinite product at $r = 1$, $r = 2$, only need be calculated in the inversion integrand. Assuming therefore that the zeros of $\phi(p)$ may be determined approximately from (25) as the roots of a quadratic in p and writing these as $[\alpha_1^2]$, $[\alpha_2^2]$ the transform solution $\bar{\theta}_i(p)$ may be inverted as indicated in (26) to give

$$\theta_i(\tau) \simeq \frac{QA}{1 + VA} + \frac{\theta_o(B_s - C[\alpha_1^2])\,[\alpha_2^2]}{(1 + VA)B_s([\alpha_2^2] - [\alpha_1^2])} \times \left\{1 + \frac{B_iV(B_s - C[\alpha_1^2])(\cos[\alpha_1] + (B_o/[\alpha_1])\sin[\alpha_1])}{B_oC(B_i + B_s + V)(\beta - [\alpha_1^2])}\right\}$$
$$\times \exp(-[\alpha_1^2]\tau) + \frac{\theta_o(B_s - C[\alpha_2^2])[\alpha_1^2]}{(1 + VA)B_s([\alpha_1^2] - [\alpha_2^2])} \times \left\{1 + \frac{B_iV(B_s - C[\alpha_2^2])(\cos[\alpha_2] + (B_o/[\alpha_2])\sin[\alpha_2])}{B_oC(B_i + B_s + V)(\beta - [\alpha_2^2])}\right\}$$
$$\times \exp(-[\alpha_2^2]\tau)$$
$$+ \frac{\theta_oVB_s^2}{C\beta(B_i + B_s + V)^2} \times \left\{1 + \frac{B_i(B_s - C\beta)^2(B_i + B_s + V)\,[\cos\sqrt{(\beta)} + (B_o/\sqrt{\beta})\sin\sqrt{\beta}]}{B_o\,B_s^3(1 + VA)(1 - \beta/[\alpha_1^2])\,(1 - \beta/[\alpha_2^2])}\right\}$$
$$\times \exp(-\beta\tau). \tag{27}$$

In either form of solution, (23) or (27), the steady state term $QA/(1 + VA)$ is equivalent to $q/(H + v)$, where H, v denote respective the initial conduction and ventilation rates of heat loss from the enclosure per unit difference between the inside and outside temperatures. The first term or constant therefore represents the initial inside temperature excess over the outside temperature. Writing this as $[\theta_i(0) - \theta_o]$ and substituting into (27), the approximate solution passes into the more convenient form:

$$\frac{\theta_i(0) - \theta_i(\tau)}{\theta_o} \simeq 1 - \frac{(B_s - C[\alpha_1^2])[\alpha_2^2]}{B_s(1 + VA)([\alpha_2^2] - [\alpha_1^2])} \times \left\{1 - \frac{VB_i(B_s - C[\alpha_1^2])(\cos[\alpha_1] + (B_o/[\alpha_1])\sin[\alpha_1])}{CB_o(B_i + B_s + V)(\beta - [\alpha_1^2])}\right\}$$
$$\times \exp(-[\alpha_1^2]\tau) + \frac{(B_s - C[\alpha_2^2])[\alpha_1^2]}{B_s(1 + VA)([\alpha_2^2] - [\alpha_1^2])} \times \left\{1 + \frac{VB_i(B_s - C[\alpha_2^2])(\cos[\alpha_2] + (B_o/[\alpha_2])\sin[\alpha_2])}{CB_o(B_i + B_s + V)(\beta - [\alpha_2^2])}\right\}$$
$$\times \exp(-[\alpha_2^2]\tau) - \frac{VB_s^2}{C\beta(B_i + B_s + V)^2}$$
$$\times \left\{1 + \frac{B_i(B_s - C\beta)^2(B_i + B_s + V)\,[\cos\sqrt{(\beta)} + (B_o/\sqrt{\beta})\sin\sqrt{\beta}]}{B_oB_s^3(1 + VA)(1 - \beta/[\alpha_1^2])(1 - \beta/[\alpha_2^2])}\right\} \exp(-\beta\tau). \quad (28)$$

Particular cases follow by taking limiting values of the parameters; for example putting $V = 0$ to correspond with an unventilated enclosure having internal heat storage, (28) reduces to

$$\frac{\theta_i(0) - \theta_i(\tau)}{\theta_o} = 1 - \frac{(B_s - C[\alpha_1^2])[\alpha_2^2]\exp(-[\alpha_1^2]\tau)}{B_s([\alpha_2^2] - [\alpha_1^2])}$$
$$+ \frac{(B_s - C[\alpha_2^2])[\alpha_1^2]\exp(-[\alpha_2^2]\tau)}{B_s([\alpha_2^2] - [\alpha_1^2])} \quad (29)$$

The simplest solution applies to the case $C = 0$, $V = 0$, when

$$\frac{\theta_i(0) - \theta_i(\tau)}{\theta_o} = 1 - \frac{[\alpha_2^2]}{[\alpha_2^2] - [\alpha_1^2]}\exp(-[\alpha_1^2]\tau)$$
$$+ \frac{[\alpha_1^2]}{[\alpha_2^2] - [\alpha_1^2]}\exp(-[\alpha_2^2]\tau). \quad (30)$$

Further examination shows $[\alpha_1^2]$ to be a satisfactory approximation to α_1^2 but the method yields a poor approximation for the second root.

The results of numerical calculation given below indicate that in general $[\alpha_2^2] \gg [\alpha_1^2]$; in the region where $|p|$ is small therefore the first root $[\alpha_1^2]$ only is required in the inversion and the solution, for example, for the fractional temperature change inside a sealed enclosure having no internal heat storage approximates satisfactorily to the simple decay expression of familiar form:

$$\frac{\theta_i(0) - \theta_i(\tau)}{\theta_o} = 1 - \exp(-[\alpha_1^2]\tau). \quad (31)$$

Assuming further that the single root $[\alpha_1^2]$ in (31) may be calculated by writing $\phi(p)$ as a linear function only in p, it follows that with $C = 0$ and $V = 0$,

$$[\alpha_1^2] \simeq \frac{1}{1/B_o + 1/2}.$$

It will be clear that the approximate solution (31) defines the exponential decay in air temperature that follows from representing the cooling process simply as

$$\frac{d\theta_i(\tau)}{dt} + [\alpha_1^2]\theta_i(\tau) = 0. \quad (33)$$

From (31), when $\tau = 1/[\alpha_1^2]$, $\dfrac{\theta_i(0) - \theta_i(\tau)}{\theta_o} = 1 - 1/e$ so that $(1/[\alpha_1^2])$ is shown to be the value of τ for which the air temperature inside the unventilated enclosure has fallen by about 63 per cent of the temperature change applied externally. By analogy with current flow in a capacitative circuit consisting of a condenser, capacitance C, discharging through a series resistance R, $l^2/K[\alpha_1^2]$ may be identified as the thermal time-constant of the cooling system (33). In current flow the time constant is well known as the product RC: a similar analogous expression may be shown to apply in the present approximate calculation of heat flow. For, relative to θ_o as base temperature, the initial heat content per unit area of the wall of thickness l and volumetric specific heat ρs is given by

$$W = \rho s l \left[\frac{\theta(l,0) + \theta(0,0)}{2} - \theta_o\right]$$

$$= \frac{\rho s l}{2} q \left(\frac{2}{h_o} + \frac{l}{k}\right)$$

or

$$\frac{W}{q} = \frac{l^2}{K}\left(\frac{1}{B_o} + \frac{1}{2}\right)$$

$$= \frac{l^2}{K} \cdot \frac{1}{[\alpha_1^2]}. \tag{34}$$

By substituting from the relation (34) into (31) it follows that

$$\frac{\theta_i(0) - \theta_i(t)}{\theta_o} = 1 - \exp\left[-t/(W/q)\right]. \tag{35}$$

The expression (35) recognizes the time constant for the simplified case represented by (33) as the ratio (heat stored/rate of heat transmission) in the steady state.

NUMERICAL RESULTS AND DISCUSSION

To illustrate the application of the above solutions, the cooling curves of six enclosures of different construction have been calculated, for a ventilation rate, in the first instance, of two air changes per hour. By repeating the calculations for zero ventilation the transient response of the enclosure structure alone is evaluated. Table 1 sets out the various cases considered as 1, 1a, 2, 2a etc. and Table 2 summarizes the data used in the numerical calculations.

The structure type is described as heavy or light according to the weight per unit area of the external walls. Basically, three different pairs of enclosure types are considered: 1 and 2; 3 and 4; 5 and 6. Cases 1, 2 and 3, 4 differ only as regards the weight of the external wall; in all other respects they are identical. The volume enclosed in Cases 1–4 is four times that for Cases 5 and 6; the (volume/wall area) ratio is 10·5 for Cases 1–4 and 5·5 for Cases 5 and 6. The cases have been selected and grouped to demonstrate within each pair the influence of internal heat storage on the rate of cooling at the ventilation rates indicated. A comparison between pairs indicates the influence of the thermal capacity of the external wall.

Table 1. Reference code of numerical examples

Structure type	Ventilated $V \neq 0$	Not ventilated $V = 0$
Heavy, $C \neq 0$	1	1a
Heavy, $C = 0$	2	2a
Light, $C \neq 0$	3	3a
Light, $C = 0$	4	4a
Heavy, $C \neq 0$	5	5a
Heavy, $C = 0$	6	6a

Table 2. Data used in numerical examples

Case	k/l	K/l^2	c	v	h_i	h_o	h_s
1	0·750	0·053	19·2	0·398	1·43	3·33	1·43
1a	0·750	0·053	19·2	—	1·43	3·33	1·43
2	0·750	0·053	—	0·398	1·43	3·33	—
2a	0·750	0·053	—	—	1·43	3·33	—
3	0·750	0·367	19·2	0·398	1·43	3·33	1·43
3a	0·750	0·367	19·2	—	1·43	3·33	1·43
4	0·750	0·367	—	0·398	1·43	3·33	—
4a	0·750	0·367	—	—	1·43	3·33	—
5	0·723	0·053	7·34	0·210	1·43	3·33	1·43
5a	0·723	0·053	7·34	—	1·43	3·33	1·43
6	0·723	0·053	—	0·210	1·43	3·33	—
6a	0·723	0·053	—	—	1·43	3·33	—

Notes:
(i) The value selected for h_s implies that the surface areas of the enclosing wall and the enclosed mass are equal.
(ii) The values selected for v, being based on a ventilation rate of two air changes per hour, imply that the ratio of the enclosed volume to the area of the enclosing wall takes the following values:

Cases 1, 2, 3, 4—10·48
Cases 5, 6 — 5·53.

It is convenient to calculate the cooling curves using the solutions as expressed in dimensionless notation. The curves for cases shown in Figs. 2 and 3 have been calculated using the exact form of solution and are plotted to show the change with time in the inside air temperature expressed as a fraction of the causative drop in the ambient outside temperature. In Fig. 2 the curves refer to enclosures ventilated at two air changes per hour; the corresponding curves for zero ventilation are shown in Fig. 3. A horizontal line drawn

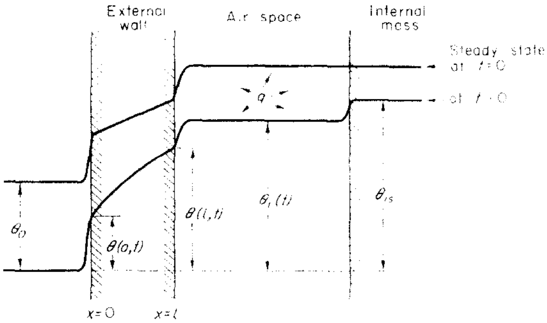

FIG. 1. Schematic representation of temperature change in a heated enclosure.

Table 3. Summary of values of time constant, T, for all cases determined by the exact solution

Structure	Time constant (h)			
	Ventilated		Not ventilated	
	Case	$V \neq 0$ T (h)	Case	$V = 0$ T (h)
Heavy, $C \neq 0$	1	22·5	1a	57
Heavy, $C = 0$	2	7·2	2a	13·7
Light, $C \neq 0$	3	14·5	3a	44·3
Light, $C = 0$	4	1·0	4a	2·0
Heavy, $C \neq 0$	5	20·0	5a	30·5
Heavy, $C = 0$	6	9·1	6a	13·7

through the vertical scale at $[\theta_i(0) - \theta_i(t)]/\theta_o = 1 - e^{-1}$, or 0·63 approximately intersects each curve at the corresponding value of the thermal time constant, T, which is read off along the time scale. A summary of the time constants obtained from the curves is given in Table 3.

An indication of the influence of ventilation on the rate at which an enclosure cools can be obtained from a comparison of the corresponding curves in Figs. 2 and 3 and, quantitatively, from the respective values of thermal time constant. In Cases 1 to 4, the time constant of the sealed enclosure is roughly two to three times that of an identical one ventilated at two air changes per hour. For a smaller structure, represented by Cases 5 and 6, the corresponding increase is less, being about 50 per cent for the same air change rate. The curves in Fig. 2 indicate an instantaneous drop in the inside air temperature. In these cases the applied step-function drop in the outside air temperature is transmitted directly by the ventilating air to the inside air, and θ_i responds accordingly in a manner influenced by the thermal capacity of the structure. In those cases with $C \neq 0$, the heat stored internally transfers to the inside air as its temperature begins to fall, thereby helping to offset the cooling influence of the cooler outside air as it enters the heated enclosure. In all cases the effect of the internal mass is to increase the time constant; with the values chosen for the illustrative examples, T is increased by many times the corresponding value for the enclosure when empty.

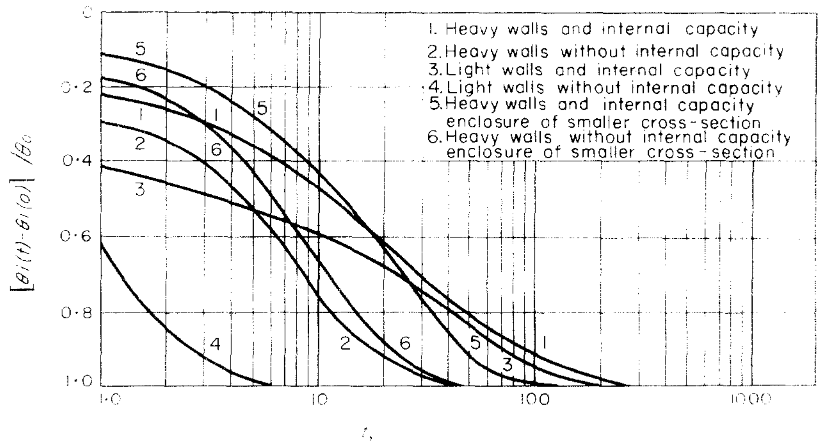

FIG. 2. Transient cooling of enclosures ventilated at two air-changes per hour

A further result is the influence of the thermal capacity of the external wall. The air enclosed behind a lightweight wall cools more quickly than in a similar enclosure with a heavy cladding, as indicated by the appropriate comparison of values of T in Table 3. For the cases considered this effect, though large, is less than that of the internal heat storage in its influence on the cooling rate of the inside air. The effect of internal heat storage will become even greater as the value of h_s increases. The present numerical results assume that $h_s = h_i$ and that the surface area of the internal mass is the same as that of the enclosure wall; if it were possible to arrange circumstances so that the value of $1/h_s$ were negligibly small, which is equivalent to writing $\theta_i(t) = \theta_{is}(t)$ for all t, the cooling interval from initial time to T would be lengthened, reaching maximum values of 24 and 46 h in the lightweight structures, Cases 3 and 3a, respectively. With the exception of Case 1, which gives an increase in T of about 40 per cent, the heavily clad structures are found to be little affected by such an extreme change in the value of the interior surface transfer coefficient h_s.

The temperature of the internal mass may be of interest. Following the statement in the formulation of the problem above that the internal mass is at a uniform temperature θ_{is} throughout its bulk, an expression for the time variation of θ_{is} is obtained from (17) by convolution, giving a solution of the form

$$\theta_{is}(\tau) = \theta_{is}(0) \exp(-[B_s/C]\tau) + \frac{B_s}{C} \int_0^\tau \exp(-[B_s/C][\tau - \tau']) \, . \, \theta_i(\tau') \, d\tau'.$$

The exact solution, equation (23), for $\theta_i(\tau)$ is lengthy and the determination of the thermal time constant T using this form of expression involves considerable numerical work. For certain of the cases considered simple expressions for the time variation of θ_i follow from using the first and second roots calculated by the approximate method indicated in (25) *et seq.* Equation (29), for example, is an approximate solution for the Cases 1a, 3a and 5a ($C \neq 0$, $V = 0$) and the results in Table 4 demonstrate the efficiency of the method. For relatively small values of time the error is large, but it diminishes with increasing values of time, and is negligible when T is reached. Similar results are obtained with the approximate solution (31) applied to Cases 2a, 4a and 6a ($C = 0$, $V = 0$). Cooling curves calculated with the approximate result (31) are shown in Fig. 3. The quantity $l^2/K[\alpha_1^2]$ has been recognized in the analysis, (35), as the ratio (heat stored/rate of heat loss from the enclosure) in the steady state and, by definition, is the time constant of the system $C = 0$, $V = 0$, as represented by (31). Values of this ratio, denoted W/q, are found to agree exactly with the corresponding values of the thermal time constant T calculated from the exact solution for

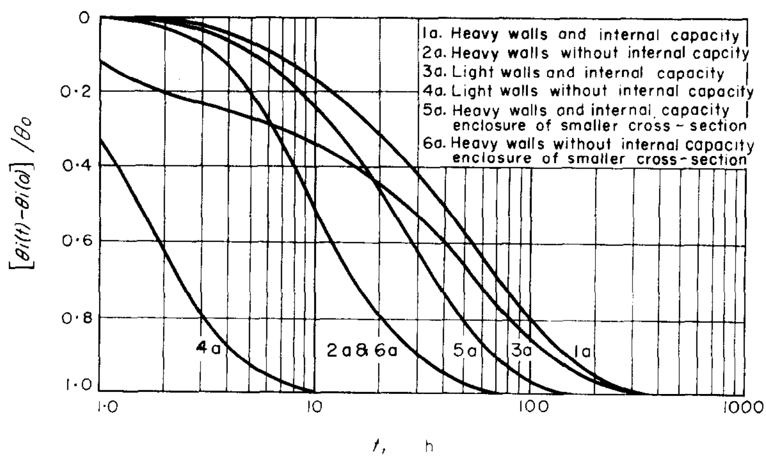

FIG. 3. Transient cooling of unventilated enclosures.

Table 4. Comparison of results calculated by exact and approximate methods: Cases 1a, 3a and 5a ($C \neq 0$, $V = 0$)*

Time (h)	$\dfrac{(B_s - C[\alpha_1^2])[\alpha_2^2]\exp(-[\alpha_1^2]\tau)}{B_s([\alpha_2^2] - [\alpha_1^2])}$	$\dfrac{(B_s - C[\alpha_2^2])[\alpha_1^2]\exp(-[\alpha_2^2]\tau)}{B_s([\alpha_2^2] - [\alpha_1^2])}$	$[\theta i(0) - \theta i(\tau)]/\theta_0$; Approx.	Exact
Case 1a				
10	0·7753	0·0278	0·1969	0·1666
20	0·6622	0·0084	0·3294	0·3067
40	0·4831	0·0008	0·5161	0·4971
100	0·1875	...	0·8125	0·8041
200	0·0387	...	0·9613	0·9592
Case 3a				
1	0·7756	0·0943	0·1301	0·1219
2	0·7626	0·0421	0·1953	0·2056
4	0·7371	0·0084	0·2545	0·2598
10	0·6655	0·0001	0·3344	0·3377
20	0·5615	...	0·4385	0·4410
40	0·3995	...	0·6005	0·6018
100	0·1440	...	0·8560	0·8561
Case 5a				
4	0·9794	−0·0758	0·0964	0·0609
10	0·7837	−0·0350	0·2413	0·2380
20	0·5591	−0·0097	0·4506	0·4601
40	0·2773	−0·0007	0·7234	0·7283
100	0·0339	...	0·9661	0·9677

* See equation (29).

these particular cases. The approximate form of solution (31) has been extended to the Cases 2, 4 and 6 ($C = 0$, $V \neq 0$) with q in the ratio W/q representing the total heat loss including the amount due to ventilation, and gives the curves shown in Fig. 4. The pattern of results is similar to that of the previous group with $C = 0$, $V = 0$ and again the time constants are found to agree closely with the exact values (see Table 5).

The fundamental significance of the ratio W/q in transient heat flow appears to have been recognized first by Reiher [4]. Esser and Krischer [5] described the cooling of a plane wall with an equation of the form

$$\theta(x, t) = \theta(x, t_u) \exp (t - t_u)/\psi(W/q)$$

where t_u is defined as the time taken for the cooling process to spread through the whole wall, and ψ is a numerical value depending on the values of t_u. When the cooling process is rapid, ψ approaches unity. The application of this simple ratio of steady state terms to the cooling and warming of buildings is discussed by Bruckmeyer [3].

Table 5. W/q, ratio of heat stored to heat transmitted compared with T, the exact value of the time constant

Structure type	Ventilated $V \neq 0$ Case	W/q (h)	T (h)	Not ventilated $V = 0$ Case	W/q (h)	T (h)
Heavy, $C = 0$	2	7·1	7·2	2a	13·7	13·7
Light, $C = 0$	4	1·0	1·0	4a	2·0	2·0
Heavy, $C \neq 0$	6	9·0	9·1	6a	13·7	13·7

For the most general Cases 1, 3 and 5, the approximate solution (28) was found to be unsatisfactory for all values of time. It appears from the present numerical results that it is necessary to use the exact solution for calculating $\theta_i(\tau)$ for enclosures that contain internal heat storage and are ventilated.

In each of the cases so far considered the wall

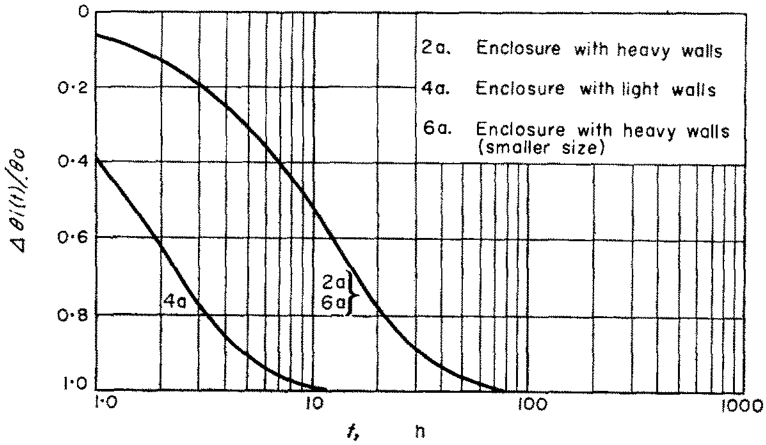

FIG. 4. Transient cooling curves based on W/q, i.e. (heat stored/heat loss) for unventilated enclosures.

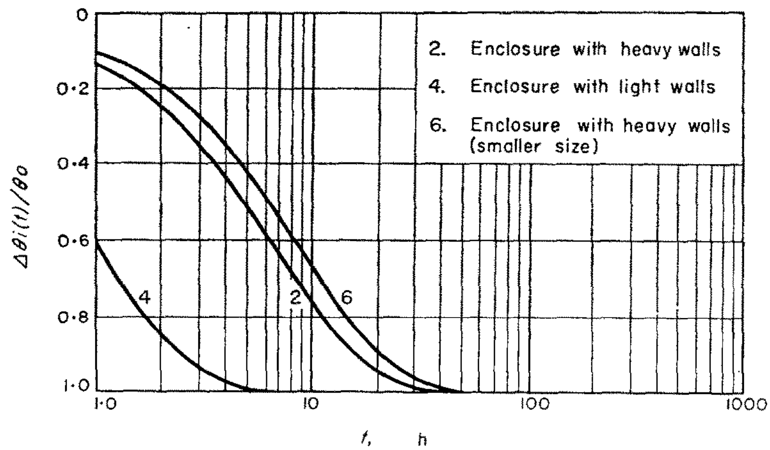

FIG. 5. Transient cooling curves based on W/q, i.e. (heat stored/heat loss) for ventilated enclosures.

of the enclosure is assumed to be a homogeneous slab. The analysis indicates that the thermal capacity of the wall has a significant effect on the rate of temperature change of the inside air. In the case of an enclosure without internal heat storage the change in θ_i after a minimum time interval is determined completely by the simple ratio (heat stored/rate of heat transmission) in the steady state. The simplicity of this result could be especially useful in the practical application of this type of solution and it is worthwhile enquiring whether it extends to other, less simple, forms of wall structure. An extension of the analysis, reported in Appendix 1, confirms that W/q is the time-constant also for an enclosure wall of more complex construction consisting of two leaves of material of different thickness, conductivity and diffusivity, separated by a sealed airspace of uniform width.

CONCLUSIONS

Solutions have been obtained to a problem in transient heat flow, defining the change with time of the air temperature inside a uniformly heated enclosure following a step-function change in the ambient outside temperature. To illustrate the application of these formulae for numerical purposes, cooling curves have been calculated for different enclosures representing a wide range of conditions of ventilation, internal heat storage and thermal capacity of the wall of the enclosure. The various cases are compared on the basis of the thermal time constant. This property of the enclosure is defined as the interval from initial time, when the system is in the steady state, to the instant when the change in the inside air temperature reaches $(1 - e^{-1})$ (where e is the base of natural logarithms) of the sudden change in the outside temperature. For a sealed enclosure without internal heat storage the time constant may be calculated most simply from the properties of the external wall as the ratio W/q, denoting (heat stored/rate of heat transmission) in the steady state. The values of W/q agree almost exactly with values of the time constant determined from the exact solution. Good agreement between these quantities is also found for the ventilated enclosure without internal heat storage with q, in this case, denoting the sum of the conduction and ventilation heat loss per unit area of wall. It is also shown that the ratio W/q has the same significance for both homogeneous and composite structures forming the wall of the enclosure.

ACKNOWLEDGEMENT

The work described in this paper forms part of the research programme of the Building Research Board of the Department of Scientific and Industrial Research and is published by permission of the Director of Building Research.

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APPENDIX 1

The above analysis is repeated with a composite structure replacing the single homogeneous slab representing the external wall. This composite wall consists of two homogeneous slabs of different material, and an airspace, unventilated and of uniform width, sandwiched between them. The two slabs, denoted regions 1 and 2, have properties (l_1, k_1, K_1), $[(l_2 - l_1), k_2, K_2]$ and temperatures $\theta_1(x, t)$, $0 < x < l_1$; $\theta_2(x, t)$, $l_1 < x < l_2$. At $x = l_1$ the surfaces exchange heat across the cavity at a rate h times their temperature difference per unit area, x being invariate in the cavity. The heated enclosure is ventilated but has no internal heat storage.

The problem, formulated mathematically, is as follows:

$$K_1 \frac{\partial^2\theta_1(x, t)}{\partial x^2} = \frac{\partial\theta_1(x, t)}{\partial t}, \quad (0 < x < l_1, \; t > 0) \qquad (1.1)$$

$$-k_1 \frac{\partial\theta_1(x, t)}{\partial x} = -h_o\theta_1(x, t), \quad (x = 0, \; t > 0) \qquad (1.2)$$

$$-k_1 \frac{\partial\theta_1(x, t)}{\partial x} = h[\theta_1(x, t) - \theta_2(x, t)], \quad (x = l_1, \; t > 0) \qquad (1.3)$$

$$K_2 \frac{\partial^2\theta_2(x, t)}{\partial x^2} = \frac{\partial\theta_2(x, t)}{\partial t}, \quad (l_1 < x < l_2, \; t > 0) \qquad (1.4)$$

$$-k_2 \frac{\partial\theta_2(x, t)}{\partial x} = h[\theta_1(x, t) - \theta_2(x, t)], \quad (x = l_1, \; t > 0) \qquad (1.5)$$

$$-k_2 \frac{\partial\theta_2(x, t)}{\partial x} = h_i[\theta_2(x, t) - \theta_i(t)], \quad (x = l_2, \; t > 0) \qquad (1.6)$$

$$q = h_i[\theta_i(t) - \theta_2(x, t)] + v\theta_i(t), \quad (x = l_2, \; t > 0) \qquad (1.7)$$

$$\theta_1(x, 0) = \theta_o + \frac{(1/h_o + x/k_1)}{1 + vR_{i\,.\,o}} q, \quad (0 < x < l_1, \; t = 0)$$

$$\theta_2(x, 0) = \theta_o \frac{(1/h_o + l_1/k_1 + 1/h + (x - l_1)/k_2)}{1 + vR_{i\,.\,o}}, \quad (l_1 < x < l_2, \; t = 0)$$

where

$$R_{t\,.\,o} = 1/h_o + l_1/k_1 + 1/h + (l_2 - l_1)/k_2 + 1/h_t.$$

Let $\bar{\theta}_r(x, p)$, $(r = 1, 2)$, denote the Laplace transform of $\theta_r(x, t)$. Applying the usual Laplace transform procedure to (1.1) through (1.7), the function $\bar{\theta}_r(x, p)$ satisfies the following transformed equations

$$K_1 \frac{d^2\bar{\theta}_1(x, p)}{dx^2} - p\bar{\theta}_1(x, p) + \frac{(1/h_o + x/k_1)}{1 + vR_{t\,.\,o}} q + \theta_o = 0, \; (0 < x < l_1, t > 0) \quad (1.8)$$

$$k_1 \frac{d\bar{\theta}_1(x,p)}{dx} = h_0\, \bar{\theta}_1(x, p), \; (x = 0, t > 0) \quad (1.9)$$

$$k_1 \frac{d\bar{\theta}_1(x,p)}{dx} = h[\bar{\theta}_2(x, p) - \bar{\theta}_1(x, p)], \quad (x = l_1, t > 0) \quad (1.10)$$

$$K_2 \frac{d^2\bar{\theta}_2(x, p)}{dx^2} - p\bar{\theta}_2(x, p) + \frac{(1/h_o + l_1/k_1 + 1/h + (x - l_1)/k_2)}{1 + vR_{t\,.\,o}} q + \theta_o = 0, \; (l_1 < x < l_2, t > 0) \quad (1.11)$$

$$k_2 \frac{d\bar{\theta}_2(x,p)}{dx} = h[\bar{\theta}_2(x, p) - \bar{\theta}_1(x, p)], \quad (x = l_1, t > 0) \quad (1.12)$$

$$k_2 \frac{d\bar{\theta}_2(x,p)}{dx} = h_t[\bar{\theta}_1(p) - \bar{\theta}_2(x, p)], \quad (x = l_2, t > 0) \quad (1.13)$$

$$\frac{q}{p} = h_t[\bar{\theta}_t(p) - \bar{\theta}_2(x, p)] + v\bar{\theta}_t(p), \quad (x = l_2, t > 0). \quad (1.14)$$

The solution of (1.8) and (1.11) subject to (1.9) (1.10) and (1.12), (1.13) respectively is

$$\left.\begin{aligned}\bar{\theta}_1(x, p) = {} & \frac{\theta_o}{p} + \frac{R_1(x)}{(1 + vR_{t\,.\,o})} \cdot \frac{q}{p} - \frac{a_o\theta_o}{pf(p)}\left[\sqrt{(p)} \cosh \sqrt{\left(\frac{p}{K_1}\right)} (l_1 - x) + a \sinh \right. \\ & \left. \sqrt{\left(\frac{p}{K_1}\right)} (l_1 - x)\right] + \frac{a_o a a_1\theta_o}{pf(p)g(p)} \left[\sqrt{(p)} \cosh \sqrt{\left(\frac{p}{K_2}\right)} (l_2 - l_1) + \frac{a_2 v}{h_t + v} \sinh \right. \\ & \left. \sqrt{\left(\frac{p}{K_2}\right)} (l_2 - l_1)\right] \times \left[\sqrt{(p)} \cosh \sqrt{\left(\frac{p}{K_1}\right)} x + a_o \sinh \sqrt{\left(\frac{p}{K_1}\right)} x\right] + \frac{a a_2\theta_o}{pf(p)g(p)} \cdot \frac{v}{(h_t + v)} \\ & \times \left[(a_o + a) \sqrt{(p)} \cosh \sqrt{\left(\frac{p}{K_1}\right)} l_1 + (p + a_o a) \sinh \sqrt{\left(\frac{p}{K_1}\right)} l_1\right] \left[p \cosh \sqrt{\left(\frac{p}{K_1}\right)} x \right. \\ & \left. + a_o \sinh \sqrt{\left(\frac{p}{K_1}\right)} x\right], \; (0 \leqslant x \leqslant l_1)\end{aligned}\right\} (1.15)$$

$$\left.\begin{aligned}\bar{\theta}_2(x, p) = {} & \frac{\theta_o}{p} + \frac{R_2(x)}{(1 + vR_{t\,.\,o})} \cdot \frac{q}{p} + \frac{a_o a_1\theta_o}{pg(p)}\left[\sqrt{(p)} \cosh \sqrt{\left(\frac{p}{K_2}\right)} (l_2 - x) \right. \\ & \left. + \frac{a_2 v}{h_t + v} \sinh \sqrt{\left(\frac{p}{K_2}\right)} (l_2 - x)\right] + \frac{a_2\theta_o}{pg(p)} \cdot \frac{v}{(h_t + v)}\left\{\left[(a_o + a)\sqrt{(p)} \cosh \sqrt{\left(\frac{p}{K_1}\right)} l_1 \right.\right. \\ & \left. + (p + a_o a) \sinh \sqrt{\left(\frac{p}{K_2}\right)} l_1\right] \times \cosh \sqrt{\left(\frac{p}{K_2}\right)} (x - l_1) + \left[a_1\sqrt{(p)} \sinh \right. \\ & \left.\left. \sqrt{\left(\frac{p}{K_1}\right)} l_1 + a_o a_1 \cosh \sqrt{\left(\frac{p}{K_1}\right)} l_1\right] \sinh \sqrt{\left(\frac{p}{K_1}\right)} (x - l_1)\right\}, \; (l_1 \leqslant x \leqslant l_2)\end{aligned}\right\} (1.16)$$

where

$$f(p) = (a_o + a)\sqrt{(p)} \cosh \sqrt{\left(\frac{p}{K_1}\right)} l_1 + (p + a_o a) \sinh \sqrt{\left(\frac{p}{K_1}\right)} l_1 \tag{1.17}$$

$$\left.\begin{aligned} g(p) = &-\Bigg\{ p^{3/2} \sinh \sqrt{\left(\frac{p}{K_1}\right)} l_1 \sinh \sqrt{\left(\frac{p}{K_2}\right)} (l_2 - l_1) + p\left[\left(a_1 + \frac{a_2 v}{h_i + v}\right) \sinh\right. \\ &\sqrt{\left(\frac{p}{K_2}\right)} l_1 \cosh \sqrt{\left(\frac{p}{K_2}\right)} (l_2 - l_1) + \left.(a_o + a) \cosh \sqrt{\left(\frac{p}{K_1}\right)} l_1 \sinh \sqrt{\left(\frac{p}{K_2}\right)} (l_2 - l_1)\right] \\ &+ \sqrt{(p)} \Bigg\{ \left(a_o a + \frac{a_1 a_2 v}{h_i + v}\right) \sinh \sqrt{\left(\frac{p}{K_1}\right)} l_1 \sinh \sqrt{\left(\frac{p}{K_2}\right)} (l_2 - l_1) \\ &+ \left[a_o a_1 + (a_o + a) \frac{a_2 v}{h_i + v}\right] \times \cosh \sqrt{\left(\frac{p}{K_1}\right)} l_1 \cosh \sqrt{\left(\frac{p}{K_2}\right)} (l_2 - l_1) \Bigg\} + \frac{a_o a_2 v}{h_i + v} \\ &\left[a \sinh \sqrt{\left(\frac{p}{K_1}\right)} l_1 \cosh \sqrt{\left(\frac{p}{K_2}\right)} (l_2 - l_1) a_1 \cosh \sqrt{\left(\frac{p}{K_1}\right)} l_1 \sinh \sqrt{\left(\frac{p}{K_2}\right)} (l_2 - l_1)\right] \Bigg\} \end{aligned}\right\} \tag{1.18}$$

$$a_o = h_o\sqrt{(K_1)}/k_1,\ a = h\sqrt{(K_1)}/k_1,\ a_1 = h\sqrt{(K_2)}/k_2,\ a_2 = h_i\sqrt{(K_2)}/k_2.$$

$$R_1(x) = 1/h_o + x/k_1,\ (0 \leqslant x \leqslant l_1)$$

$$R_2(x) = 1/h_o + l_1/k_1 + 1/h + (x - l_1)/k_2,\ (l_1 \leqslant x \leqslant l_2)$$

$$R_{i\,.\,o} = 1/h_o + l_1/k_1 + 1/h + (l_2 - l_1)/k_2 + 1/h_i. \tag{1.19}$$

From (1.14)

$$\bar{\theta}_i(p) = [h_i\bar{\theta}_2(l_2, p) + q/p]/(h_i + v). \tag{1.20}$$

Substituting for $\bar{\theta}_2(l_2, p)$ from (1.16) in (1.20) and applying the inversion theorem to the resulting expression for $\bar{\theta}_i(p)$ gives the exact function of time thus

$$\left.\begin{aligned} \theta_i(t) = &\frac{(1/h_o + l_1/k_1 + 1/h + (l_2 - l_1)/k_2)}{1 + vR_{i.o}} q + \frac{q}{h_i + v} + \frac{2h_i\theta_o}{h_i + v} \sum_{n=1}^{\infty} \frac{\exp(-\alpha_n^2 t)}{\alpha_n G(\alpha_n)} \\ &\times \left\{ a_o a_1 + \frac{a_2 v}{h_i + v}\left[(a_o + a)\cos b_1\alpha_n + \left(\frac{a_o a}{\alpha_n} - \alpha_n\right)\sin b_1\alpha_n\right] \cos b_2\alpha_n \right. \\ &\left. + (a_o \cos b_1\alpha_n - \alpha_n \sin b_1\alpha_n)\frac{a_1}{\alpha_n} \sin b_2\alpha_n \right\} \end{aligned}\right\} \tag{1.21}$$

where the summation extends over the positive roots, α_n of

$$\left(\alpha^2 - a_o a - \frac{a_1 a_2 v}{h_i + v}\right) \tan b_1\alpha \,.\, \tan b_2\alpha - \left[\alpha\left(a_1 + \frac{a_2 v}{h_i + v}\right) - \frac{a_o a a_2 v}{(h_i + v)\alpha}\right] \tan b_1\alpha$$

$$- \left[\alpha(a_o + a) - \frac{a_o a_1 a_2 v}{(h_i + v)\alpha}\right] \tan b_2\alpha + a_o a_1 + \frac{(a_1 + a)a_2 v}{h_i + v} = 0,$$

and

$$G(\alpha_n) = \left\{\frac{a_o a a_2 v}{(h_i + v)} + \frac{1}{\alpha_n} a_1 + \frac{a_2 v}{h_i + v} + \frac{b_1}{\alpha_n}\left[a_o a_1 + \frac{(a_o + a)a_2 v}{h_i + v}\right] + \frac{b_2}{\alpha_n}\, a_o a + \frac{a_1 a_2 v}{h_i + v} - \alpha_n b_2\right\}$$

$$\sin b_1\alpha_n \cos b_2\alpha_n + \left\{\frac{a_o a_1 a_2 v}{(h_i + v)} + \frac{b_1}{\alpha_n}\left(a a_o + \frac{a_1 a_2 v}{h_i + v}\right) + \frac{b_2}{\alpha_n}\left[a_o a_1 + \frac{(a_o + a)a_2 v}{h_i + v}\right] + \frac{(a_o + a)}{\alpha_n}\right.$$

$$\left. - b_1\alpha_n\right\} \times \cos b_1\alpha_n \sin b_2\alpha_n - \left\{2 + \left[\frac{a_1 h_i + (a_1 + a_2)v}{(h_i + v)}\right] b_2 + (a_o + a)b_1 - \frac{a_o a a_2 v}{\alpha_n^2(h_i + v)}\, b_2\right.$$

$$\left. - \frac{a_o a a_2 v}{\alpha_n^2(h_i + v)} b_1\right\} \sin b_1\alpha_n \sin b_2\alpha_n + \left\{\left[\frac{a_1 h_i + (a_1 + a_2)v}{h_i + v}\right] b_1 + (a_o + a)b_2 - \frac{a_o a a_2 v}{\alpha_n^2(h_i + v)}\, b_1\right.$$

$$\left. - \frac{a_o a a_2 v}{\alpha_n^2(h_i + v)} b_2\right\} \times \cos b_1\alpha_n \cos b_2\alpha_n$$

with $b_1 = l_1/\sqrt{K_1}$, $b_2 = (l_2 - l_1)/\sqrt{K_2}$.

Application of the inversion procedure indicated in (25) through (27) yields, for the present problem, an approximate form of solution similar to that obtained above for the enclosure with a homogeneous slab representing the wall. For, expanding $g(p)/\sqrt{p}$ from (1.18) to a linear term in p and putting $v = 0$ it may be shown that

$$\frac{\theta_i(0) - \theta_i(t)}{\theta_o} \simeq 1 - \exp(-\alpha_1^2 t),\ (t \text{ large})$$

where

$$\alpha_1^2 = a_o a_1\{a_1 b_1 + (a_o + a)b_2 + a_o a b_1 b_2 + (a_o a_1/2)[b_1^2 + b_2^2]\}^{-1}. \tag{1.22}$$

The heat stored in the wall per unit area in the initial steady state relative to θ_o as base temperature is

$$W = \rho_1 s_1 l_1 \left[\frac{\theta_1(0, 0) + \theta_1(l_1, 0)}{2} - \theta_o\right] + \rho_2 s_2(l_2 - l_1)\left[\frac{\theta_2(l_1, 0) + \theta_2(l_2, 0)}{2} - \theta_o\right]$$

$$= \rho_1 s_1 l_1(\theta_i - \theta_o)\left(\frac{1/h_o + l_1/2k_1}{R_{i\,.\,o}}\right) + \rho_2 s_2(l_2 - l_1)(\theta_i - \theta_o)\left[\frac{1/h_o + l_1/k_1 + 1/h + (l_2 - l_1)/2k_2}{R_{i\,.\,o}}\right].$$

Assuming no ventilation, heat transmission is by conduction only through the wall so that in the initial steady state

$$q = (\theta_i - \theta_o)/R_{i\,.\,o}.$$

Composing the ratio (heat stored/rate of heat transmission) in the steady state gives

$$\frac{W}{q} = \rho_1 s_1 l_1\left(\frac{1}{h_o} + \frac{l_1}{2k_1}\right) + \rho_2 s_2(l_2 - l_1)\left(\frac{1}{h_o} + \frac{l_1}{k_1} + \frac{1}{h} + \frac{l_2 - l_1}{2k_2}\right), \tag{1.23}$$

and introducing the relationships (1.19) into (1.23) leads to the result

$$\frac{W}{q} = \frac{1}{a_1 a_o}\left\{\frac{a_1 l_1}{\sqrt{K_1}} + (a_o + a)\frac{(l_2 - l_1)}{\sqrt{K_2}} + a_o a\frac{l_1}{\sqrt{K_1}} \cdot \frac{(l_2 - l_1)}{\sqrt{K_2}} + \frac{a_o a_1}{2}\left[\frac{l_1^2}{K_1} + \frac{(l_2 - l_1)^2}{K_2}\right]\right\}, \tag{1.24}$$

where the quantity on the r.h.s. will be recognized as $1/[\alpha_1^2]$, (1.22).

Résumé—On fait une étude théorique de la variation, en régime transitoire, de la température de l'air à l'intérieur d'une enceinte à laquelle on fournit un flux de chaleur constant, la température extérieure subissant une variation discontinue. Dans un petit nombre de cas particuliers on a comparé par le calcul l'effet de la ventilation, du stockage intérieur de la chaleur, des propriétés thermiques des parois de l'enceinte sur le taux de variation de la température intérieure. On montre que, dans certaines conditions, le rapport (chaleur stockée/taux de chaleur perdu par l'enceinte) est, en régime permanent, le paramètre qui détermine la variation de la température intérieure.

Zusammenfassung—Die kurzzeitigen Veränderungen der Lufttemperatur in einem Hohlraum, in dem gleichmässig Wärme erzeugt wird, dessen Umgebung aber einem schrittweisen Temperaturwechsel unterliegt, wurden theoretisch untersucht. Die Einflüsse der Ventilation, der inneren Wärmespeicherung und der thermischen Eigenschaften der Hohlraumwände auf die Temperaturänderung im Hohlraum liessen sich durch numerische Berechnungen an Hand ausgewählter Biespiele bestimmen. Unter gewissen Bedingungen stellt das Verhältnis (gespeicherte Wärme/zeitlich abgegebene Wärme) im stationären Fall den bestimmenden Parameter für die Änderung der Innentemperatur dar.

Аннотация—Теоретически исследован нестационарный температурный процесс в камере с равномерным распределением тепла, поступающего в неё путем скачкообразного изменения внешней температуры. Для некоторый отдельных случаев с помощью численного расчета выявляются зависимости охлаждения, внутреннего запаса тепла и термических свойств стенок камеры от скорости изменения внутренней температуры. Показано, что в определенных условиях отношение (накопленного тепла к телу, теряемому камерой) в установившемся состоянии представляет собой параметр, который определяет изменение внутренней температуры.